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Entropy Numbers of Vector-Valued Diagonal Operators

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Let $I_\alpha : l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})$ be a diagonal operator assigning to vector-coordinate $x_j \in l_p^{2^j}$ the vector-value $j^{-\alpha}x_j$. We prove the estimates of entropy numbers of I_α. The results confirm the conjecture stated in a recent paper by Cobos and Kühn, and extend their results to quasi-Banach setting.

Key Words: diagonal operator; entropy numbers.

1. INTRODUCTION

Let us consider the space of sequences $l_q(l_p^{2^j})$ where $0 < p \leqslant \infty, \; 0 < q \leqslant \infty$, and the (quasi)-norm defined by the formula

$$\left\{ \sum_{j=1}^{\infty} \left(\sum_{m=1}^{2^j} |a_{jm}|^p \right)^{q/p} \right\}^{1/q}$$

with the usual extension for $p = \infty$ and $q = \infty$. We study entropy numbers of the diagonal operator $I_\alpha : l_q(l_p^{2^j}) \to l_q(l_\infty^{2^j})$ which assigns to each element $a_j \in l_p^{2^j}$ the element $j^{-\alpha}a_j$. In their recent paper, Cobos and Kühn [1] proved two-sided estimates for the entropy numbers of I_α, and gave interesting applications to embedding theorems of Besov spaces into generalized Lipschitz spaces.

In this paper, we improve the lower estimates of [1], and confirm the conjectured order of the entropy numbers. We also give a new proof of upper estimates, which extends the upper estimates of [1] to the case of the quasi-Banach spaces.

Let us restate the definitions (see, for example [5]). Suppose K is a compact set in a normed space Y, $B(Y)$ is the closed unit ball.

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The entropy numbers e_n are defined by

$$e_n(K;Y) := \inf\left\{ \varepsilon \colon \exists \{x_1, \dots, x_{2^n}\} \colon K \subset \bigcup_{j=1}^{2^n} (x_j + \varepsilon B(Y)) \right\},$$

where the infimum is taken over all ε such that K can be covered by 2^n balls $\varepsilon B(Y)$ of radius ε.

If $T : X \to Y$ is an operator from space X to space Y, then we can speak about entropy numbers $e_n(T; X \to Y)$ of operator T, defining them as

$$e_n(T; X \to Y) := e_n(TB(X); Y).$$

We will use the observation of [1] that the entropy numbers of I_α are equal to the entropy numbers of the embedding operator I from the weighted space $l_q(j^\alpha l_p^{2^j})$ with the (quasi)-norm

$$\left\{ \sum_{j=1}^{\infty} \left(j^\alpha \left(\sum_{m=1}^{2^j} |a_{jm}|^p \right)^{1/p} \right)^q \right\}^{1/q}$$

to the space $l_q(l_\infty^{2^j})$.

We write that $a_n \ll b_n$ if there exists an absolute constant C such that $a_n \leqslant C b_n$, and we write $a_n \simeq b_n$ if simultaneously $a_n \ll b_n$ and $b_n \ll a_n$.

2. MAIN RESULTS

THEOREM 2.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $\alpha > 0$. *Then*

$$e_k(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \simeq \begin{cases} k^{-\frac{1}{p}}(1 + \log k)^{\frac{2}{p} - \alpha} & \text{if } \alpha > 2/p, \\ k^{-\frac{1}{p}}(1 + \log k)^{\frac{1}{p}} & \text{if } \alpha = 2/p, \\ k^{-\frac{\alpha}{2}} & \text{if } \alpha < 2/p. \end{cases}$$

The upper estimate for $p \geqslant 1$, and the lower estimate for any $p > 0$ and $\alpha < 2/p$ were proved in [1].

The main tool is the following result due to Schütt [6] for the case of Banach spaces $1 \leqslant p \leqslant q \leqslant \infty$. It was extended recently to the quasi-Banach space in [2, 4, 7].

LEMMA 2.2. *Let* $0 < p_1 \leqslant p_2 \leqslant \infty$, *then*

$$e_k(I : l^n_{p_1} \to l^n_{p_2}) \simeq \begin{cases} 1 & \textit{if } 1 \leqslant k \leqslant \log 2n, \\ \left(\dfrac{\log(\frac{2n}{k}+1)}{k}\right)^{\frac{1}{p_1}-\frac{1}{p_2}} & \textit{if } \log 2n \leqslant k \leqslant 2n, \\ 2^{-\frac{k-1}{2n}} n^{\frac{1}{p_2}-\frac{1}{p_1}} & \textit{if } k \geqslant 2n. \end{cases}$$

We also need the estimate of the entropy numbers of the diagonal operator in the scalar case [3].

LEMMA 2.3. *Let* I_α *be the diagonal operator* $I_\alpha(x) := j^{-\alpha} x_j$, $j = 1, 2, 3, \ldots$. *Then for any* q, $0 < q \leqslant \infty$

$$e_n(I_\alpha; l_q \to l_q) \simeq \frac{1}{n^\alpha}.$$

Proof of Theorem 2.1 (*Upper Estimate*). Fix $\varepsilon = 1/n^\alpha$. Lemma 2.3 states that there exist at most 2^n elements $y \in l_q$, creating ε-net $\mathscr{E}(I_\alpha : l_q \to l_q)$ of the image I_α of the unit ball of l_q in the space l_q. Even more, all coordinates $y_j = 0$ for $j = n+1, n+2, \ldots$.

Let us now construct the ε-net $\mathscr{E}(I : l_q(j^\alpha l^{2^j}_p) \to l_q(l^{2^j}_\infty))$ for the set of x, such that $\|x : l_q(j^\alpha l^{2^j}_p)\| \leqslant 1$. Each x we write as $\{x_1, x_2, \ldots, x_j, \ldots\}$ where each x_j is an element of the finite-dimensional subspace $l^{2^j}_p$. In turn, each x_j has coordinates $\{x_{j1}, \ldots, x_{jm}, \ldots, x_{j2^j}\}$. We then construct the ε-net $\mathscr{E}(I : l_q(j^\alpha l^{2^j}_p) \to l_q(l^{2^j}_\infty))$ in the space $l_q(l^{2^j}_\infty)$. We need to consider only elements x such that $x_j = 0$ for any vector x_j with $j \geqslant n$.

The sequence of numbers $\{j^\alpha \|x_j\|_{l_p}\}$ belongs to the unit ball of l_q. Therefore, there exists an element y from ε-net $\mathscr{E}(I_\alpha : l_q \to l_q)$ such that $\|\{\|x_j\|_p\} - \{y_j\}\|_{l_q} \leqslant \varepsilon$. Observe that $\|\{j^\alpha y_j\}\|_q \leqslant \text{Const}$. Let us take $\bar{x}_j = \frac{y_j}{\|x_j\|_p} x_j$. Then

$$\|x - \bar{x} : l_q(l^{2^j}_\infty)\| \leqslant \left\| \left\| x_j - \frac{y_j}{\|x_j\|_p} x_j \right\|_p \right\|_{l_q} \leqslant \varepsilon.$$

For each $j = 1, \ldots, m$, $m < n$ we apply Lemma 2.2 taking 2^{n_j} points of the net of the unit ball of $l^{2^j}_p$, where

$$\sum_{j=1}^{m} n_j = n.$$

This means that for each element $y \in \mathscr{E}(I_\alpha : l_q \to l_q)$ there exists a 2^n-element net in the space $l_q(l_\infty^{2^j})$, approximating x with the error at most

$$\left\{ \sum_{j=1}^{m} (y_j e_{n_j}(I : l_p^{2^j} \to l_\infty^{2^j}))^q + \sum_{j=m+1}^{n} (y_j)^q \right\}^{1/q}.$$

Taking into account that $\|\{j^\alpha y_j\}\|_q \leqslant \mathrm{Const}$, we estimate the error by

$$\sup_{1 \leqslant j \leqslant m} (j^{-\alpha} e_{n_j}(I : l_p^{2^j} \to l_\infty^{2^j}) + m^{-\alpha}).$$

Now we have to choose the numbers n_j in the optimal way.

If we put $n_j = \frac{2}{p} 2^j (\log n - j)$, $j = 1, 2, \ldots, \log n$, then applying the third line of Lemma 2.2 to

$$\sup_{1 \leqslant j \leqslant \log n} j^{-\alpha} e_{n_j}(I : l_p^{2^j} \to l_\infty^{2^j}),$$

we obtain

$$\sup_{1 \leqslant j \leqslant \log n} 2^{-j/p} j^{-\alpha} 2^{-n_j 2^{-j}} = \sup_{1 \leqslant j \leqslant \log n} 2^{j/p} j^{-\alpha} n^{-2/p} \ll n^{-1/p} \log^{-\alpha} n.$$

This estimate is sufficient for any $\alpha > 0$.

Choosing the numbers n_j in the main interval $\log n \leqslant j \leqslant m$, we consider three cases. We can take $m = n^{1/\alpha p} (\log n)^{1 - 2/\alpha p}$ in the first case, and $m = n^{1/2}$ in the second and third cases. In each case the application of the second line of Lemma 2.2 finishes the proof.

Case 1. Let $\alpha > 2/p$. Put $n_j = n (\log n)^{\alpha p - 2} j^{-(\alpha p - 1)}$.

Case 2. Let $\alpha < 2/p$. Put $n_j = n j^{-(\alpha p - 1)} m^{\alpha p - 2}$.

Case 3. Let $\alpha = 2/p$. Put $n_j = n j^{-1} (\log n)^{-1}$.

The upper estimate is proved. ∎

We need the following statement, which supposedly is well known in coding theory. We could not find the exact reference, and present its proof here. In the following, $|Q|$ denotes the cardinality of the set Q, and the "interval" $[1, M]$ means the set $\{1, 2, \ldots, M\}$.

Lemma 2.4. *Let us consider a "brick" $\Pi \subset \mathbb{Z}^m$, $\Pi = [1, M]^m$, where M is a natural number. Let $s < m$. Then there exists a set $Q \subset \Pi$ such that*

(1) $|Q| \geqslant \frac{M^{m-s}}{\binom{m}{s}}$.

(2) $|\{j\colon x_j \neq y_j\}| > s$ *for any two elements* $x, y \in Q$.

Proof. We introduce the Hamming distance between two points $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ of $\mathbb{Z}^m$. The Hamming distance is given by

$$H(x, y) := |\{j\colon x_j \neq y_j\}|.$$

Set Π contains M^m elements. Let Q be a *maximal* subset of Π such that for any elements $x, y \in Q$, $H(x, y) > s$. If $x \in Q$ then for at most $M^s\binom{m}{s}$ points $y \in \Pi$, $H(x, y) \leqslant s$. Since Q is the maximal subset

$$|Q| M^s \binom{m}{s} \geqslant M^m$$

or $|Q| \geqslant \frac{M^{m-s}}{\binom{m}{s}}$.

The lemma is proved. ∎

Proof of Theorem 2.1 (*Lower Estimates*). Take the blocks of vectors with numbers from n to $2n$ and let $k = 2^{n/2}$. By Lemma 2.2 for each block

$$e_{2^{n/2}}(j^{-\alpha} I : l_p^{2^j} \to l_\infty^{2^j}) \gg n^{-\alpha} \left(\frac{\log(2^{n-n/2}+1)}{2^{n/2}} \right)^{1/p} \simeq \frac{n^{1/p-\alpha}}{2^{n/2p}}, \qquad n \leqslant j \leqslant 2n.$$

Let us denote this number by ε. Then in each block we can take $2^{2^{n/2}}$ ε-separated points.

We can now apply Lemma 2.4, taking $M = 2^{2^{n/2}}$, $m = n$ and $s = n/2$. Then $|Q| \geqslant 2^{C(n/2)2^{n/2}}$. The norm of each element in $l_q(j^\alpha l_p^{2^j})$ is

$$\left(\sum_{j=n}^{2n} 1^q \right)^{1/q} \simeq n^{1/q}.$$

The distance between any two elements

$$\left(\sum_{n \leqslant j \leqslant 2n,\ x_j \neq y_j} \left(\frac{n^{1/p-\alpha}}{2^{n/2p}} \right)^q \right)^{1/q} \gg \frac{n^{1/p-\alpha+1/q}}{2^{n/2p}}.$$

Finally, we obtain

$$e_{2^{n/2} n/2}(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \frac{n^{1/p-\alpha}}{2^{n/2p}}$$

or

$$e_m(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \frac{\log^{2/p-\alpha} m}{m^{1/p}}. \tag{2.1}$$

For the same collection of blocks from n to $2n$ let $k = n$. Then by Lemma 2.2 for each block

$$e_n(j^{-\alpha}I : l_p^{2^j} \to l_\infty^{2^j}) \gg \frac{1}{n^\alpha}.$$

It means that in each block we have 2^n $n^{-\alpha}$-separated points. We apply again Lemma 2.4, taking $M = 2^n$, $m = n$, and $s = n/2$. Then $|Q| \gg 2^{Cn^2}$. The norm of each element in $l_q(j^\alpha l_p^{2^j})$ is $n^{1/q}$ and the distance between any two elements $\geqslant n^{1/q}n^{-\alpha}$. This gives

$$e_{n^2}(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \frac{1}{n^\alpha}$$

or

$$e_m(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \frac{1}{m^{\alpha/2}}. \tag{2.2}$$

These two estimates (2.1) and (2.2) give the necessary estimate from below, except for the case $\alpha = 2/p$.

Let us consider $\alpha = 2/p$, and take the same blocks numbered from n to $2n$. Then by Lemma 2.2 for each $n \leqslant k \leqslant 2^{n/2}$ and for each block

$$e_k(j^{-\alpha}I : l_p^{2^j} \to l_\infty^{2^j}) \gg \frac{1}{n^{1/p}k^{1/p}}.$$

Applying Lemma 2.4 with $M = 2^k$, $m = n$ and $s = n/2$ we obtain that for a subset

$$A_{n,2n} = \left\{ a : \left\{ \sum_{j=n}^{2n} (j^\alpha \|a_j\|_{l_p^{2^j}})^q \right\}^{1/q} \leqslant 1 \right\}$$

generated by blocks numbered from n to $2n$

$$e_{kn}(A_{n,2n}; l_q(l_\infty^{2^j})) \gg \frac{1}{n^{1/p}k^{1/p}}.$$

Let us take $r = c_1 \log(\log n), \ldots, \log n$. Then the last estimate implies that in a subspace generated by blocks numbered from 2^r to 2^{r+1}

$$e_{k2^r}(A_{2^r,2^{r+1}}; l_q(l_\infty^{2^j})) \gg \frac{1}{2^{r/p}k^{1/p}}$$

for any $2^r < k < 2^{2^{r-1}} 2^r$. Choose $k = \frac{n^2}{2^r}$. Then for each subspace generated by blocks numbered from 2^r to 2^{r+1}

$$e_{n^2}(A_{2^r,2^{r+1}}; l_q(l_\infty^{2^j})) \gg \frac{1}{n^{2/p}}.$$

We again apply Lemma 2.4, taking $M = 2^{n^2}$, $m = \log n$, and $s = \frac{1}{2}\log n$. Then

$$e_{n^2 \log n}(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \frac{1}{n^{2/p}}$$

or

$$e_m(I : l_q(j^\alpha l_p^{2^j}) \to l_q(l_\infty^{2^j})) \gg \left(\frac{\log m}{m}\right)^{1/p}.$$

Thus the theorem is proved. ∎

3. CONCLUDING REMARKS

The method of the above estimates used in the proof of Theorem 2.1 can be easily applied in a more general situation. For example, let $0 < p_1 < p_2 \leqslant \infty$ and consider operator $I_\alpha : l_q(j^\alpha l_{p_1}^{2^j}) \to l_q(l_{p_2}^{2^j})$. Then

$$e_k(I : l_q(j^\alpha l_{p_1}^{2^j}) \to l_q(l_{p_2}^{2^j}))$$
$$\ll \begin{cases} k^{-1/p_1+1/p_2}(1+\log k)^{2/p_1-2/p_2-\alpha} & \text{if } \alpha > 2/p_1 - 2/p_2, \\ (k^{-1}(1+\log k))^{1/p_1-1/p_2} & \text{if } \alpha = 2/p_1 - 2/p_2, \\ k^{-\alpha/2} & \text{if } \alpha < 2/p_1 - 2/p_2. \end{cases}$$

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